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On the Normal Solutions of the Boltzmann Equation with Small Knudsen Number

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A singular perturbation method is used to find the normal solutions of the Boltzmann equation with small Knudsen number. It is proved that the secular terms may be removed by improving the Hilbert expansion and the Enskog expansion.

KEY WORDS: Boltzmann equation; normal solution; singular perturbation; secular terms.

1. INTRODUCTION

The Boltzmann equation (B.E.) is basic to rarefied gas dynamics[1] and has been discussed extensively over the years. On the one hand, the relaxation in rarefied gases from a nonequilibrium to an equilibrium state is, conceptually, a central problem in statistical mechanics and in irreversible thermodynamics. On the other hand, rarefied gas dynamics contains a veriety of effects, interesting in themselves and important for applications.

In this paper we reexamine the systematics for solving the B.E. when the Knudsen number is small. We shall take it for granted that the B.E. correctly describes the physics, and that its solution under proper initial and boundary conditions exists.

Due to its complexity, it is generally out of the question to solve the B.E. exactly. The few exact solutions known, such as the equilibrium Maxwell distribution and the recently found BKW mode,[2-4] are insufficient to elucidate general relaxation behavior. One must therefore resort to approximate methods, for example, the moment method,[5] the model method,[6] and the Monte Carlo method.[7] In particular, for small Knud-

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sen numbers, the singular perturbation method is suitable, and will be discussed in detail in what follows.

The B.E. for a gas of monatomic molecules in the absence of an external force can be written[1]

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} = \frac{1}{\varepsilon} C(f, f) \tag{1}$$

$$C(f, f_1) = \int d\mathbf{w}\, d\hat{u}'\, u\sigma(\hat{u} \cdot \hat{u}', u)[f(\mathbf{w}') f_1(\mathbf{v}') - f(\mathbf{w}) f_1(\mathbf{v})] \tag{2}$$

where $f = f(\mathbf{r}, \mathbf{v}, t)$ is the single-particle distribution function, $\mathbf{v}, \mathbf{w}$ and $\mathbf{v}', \mathbf{w}'$ are velocities of a pair of molecules before and after the collision, respectively, and the corresponding relative velocities are $\mathbf{u} = \mathbf{v} - \mathbf{w} = u\hat{u}$ and $\mathbf{u}' = \mathbf{v}' = \mathbf{w}' = u\hat{u}'$ (where the caret denotes a unit vector). Furthermore, $(1/\varepsilon)\, \sigma(\hat{u} \cdot \hat{u}', u)$ is the differential cross section and ε is the Knudsen number, defined by $\varepsilon = l_r/l_0$, where l_r is the molecular mean free path and l_0 the characteristic length of the system. The normalization of f is chosen to be (with V the total volume)

$$V^{-1} \int f\, d\mathbf{v}\, d\mathbf{r} = 1 \tag{3}$$

We also choose units of length, time, and mass such that the conservation of energy can be expressed by

$$V^{-1} \int v^2 f\, f\mathbf{v}\, d\mathbf{r} = 3 \tag{4}$$

(Another condition concerning the units will be added in Section 2.) For small Knudsen number, $\varepsilon \ll 1$, we can take ε as the perturbation parameter. The form of Eq. (1) naturally suggests a singular perturbation problem.

For Maxwell molecules (whose differential cross section is inversely proportional to the relative speed of the colliding molecules), the B.E. simplifies. The product of u and σ is independent of u in this case. Defining

$$u\sigma(\hat{u} \cdot \hat{u}', u) = g_{\mathrm{M}}(\hat{u} \cdot \hat{u}') \tag{5}$$

we can then write the B.E. as

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} = \frac{1}{\varepsilon} C_{\mathrm{M}}(f, f) \tag{6}$$

with

$$C_{\mathrm{M}}(f, f_1) = \int d\mathbf{w}\, d\hat{u}'\, g_{\mathrm{M}}(\hat{u} \cdot \hat{u}')[f(\mathbf{w}') f_1(\mathbf{v}') - f(\mathbf{w}) f_1(\mathbf{v})] \tag{7}$$

Since the calculations simplify considerably with Maxwell molecules, we shall mainly, in Sections 2–5, consider that model. However, the conclusions of this paper are essentially model independent. This will be shown in Sections 6 and 7.

Hilbert first used the perturbation method to solve the B.E. with the small parameter ε.[8] He proposed that one should seek distribution functions in the form of a power series in

$$f = \sum_{j=0}^{\infty} \varepsilon^j f^{(j)} \tag{8}$$

The Hilbert expansion yields a sequence of the inviscid fluid equations (Euler equations). Consequently, it is obvious that this expansion can be correct for a finite time only. It cannot be used to discuss the relaxation of a nonequilibrium state to equilibrium. Enskog improved the Hilbert method and expanded not only f, as in Eq. (8), but also[9]

$$\frac{\partial}{\partial t} = \sum_{j=0}^{\infty} \varepsilon^j \frac{\partial_j}{\partial t} \tag{9}$$

and kept the hydrodynamic variables (mass density, local velocity, and temperature) unexpanded. As a result, the Enskog expansion yields the Euler equations, Navier–Stokes equations, Burnett equations, and super-Burnett equations in successive order of ε. The order of space derivatives of the distribution function in these equations is increasingly higher, and more and more boundary conditions are needed. However, it is in general impossible to find boundary conditions beyond those needed in the Navier–Stokes equations. This is the boundary condition difficulty of the Enskog expansion.

Both the Hilbert and the Enskog expansions are invalid in the "initial layer," the "boundary layer," and the "shock wave layer," because they lead to distribution functions that depend on time and space only through the hydrodynamic variables. Therefore the solutions of the B.E. in these three layers have to be discussed anew. A solution that is valid in time and space regions *excluding* these three layers is called a normal solution.

In the present paper we restrict ourselves to a discussion of normal solutions. A companion paper[15] will demonstrate how a natural extension of our method permits a discussion of the initial layer as well.

Cercignani[1] proposed (see also Ref. 14) an alernative to the Hilbert and the Enskog expansions. He expanded f as in Eq. (8), truncated the expansion of $\partial/\partial t$ as

$$\frac{\partial}{\partial t} = \sum_{j=0}^{N} \varepsilon^j \frac{\partial_j}{\partial t} \tag{10}$$

and assumed that the hydrodynamic variables should be expanded in powers of ε^{N+1}. His expansion reduces to the Hilbert expansion for $N=0$ and to the Enskog expansion when $N \to \infty$. The proper choice for N is, according to Cercignani, $N=1$.

In the present paper we show that macroscopic arguments are *not* necessary for a construction of the proper expansion in ε. We shall show that the B.E. *itself* gives $N=1$, under the natural requirement that no secular terms appears. As far as the expansion is concerned, our conclusion coincides with that of Cercignani. However, we also show that the corresponding expansion of the hydrodynamic variables should be in powers of ε, rather than in powers of $\varepsilon^{N+1}=\varepsilon^2$, as used by Cercignani. That removal of secular terms has this as a consequence becomes clear only when one wants to connect the normal solution to solutions through the initial and boundary layers. This will be discussed in detail in a companion paper.[15]

Since our method is based on a systematic removal of secular terms, it can be used to study any kind of relaxation process. This is its principal advantage over other procedures. In the present paper we shall, for simplicity, confine the discussion to planar geometry. The method and the conclusions are general, however, and can be extended to more complicated geometries without essential difficulties.

2. THE DOMINANT TERMS IN THE EXPANSION

Bobylev[10] found that the Fourier transform may be used to simplify the B.E. (1). He defined the generating function as

$$\varphi(\mathbf{r}, \mathbf{k}, t) = \int [\exp(-i\mathbf{k} \cdot \mathbf{v})] f(\mathbf{r}, \mathbf{v}, t)\, d\mathbf{v} \tag{11}$$

Then Eq. (1) can be written as

$$\frac{\partial \varphi}{\partial t} + i \frac{\partial^2 \varphi}{\partial \mathbf{k} \cdot \partial \mathbf{r}} = \frac{1}{\varepsilon} J(\varphi, \varphi) \tag{12}$$

where

$$\begin{aligned} J(\varphi, \varphi_1) &= \int [\exp(-i\mathbf{k} \cdot \mathbf{v})]\, C(f, f_1)\, d\mathbf{v} \\ &= \int d\mathbf{v}\, d\mathbf{w}\, d\hat{u}'\, u\sigma(\hat{u} \cdot \hat{u}', u) \\ &\quad \times [f(\mathbf{w}') f_1(\mathbf{v}') - f(\mathbf{w}) f_1(\mathbf{v})] \exp(-i\mathbf{k} \cdot \mathbf{v}) \end{aligned} \tag{13}$$

Since Eq. (13) is invariant when we interchange ($\mathbf{v}$, $\mathbf{w}$) and ($\mathbf{v}'$, $\mathbf{w}'$), and u, $\hat{u}\cdot\hat{u}'$, and $d\mathbf{u}\, d\mathbf{w}\, d\hat{u}'$ are invariant also, we have

$$J(\varphi, \varphi_1) = \int d\mathbf{v}\, d\mathbf{w}\, f_1(\mathbf{v})\, f(\mathbf{w}) \exp\left[-\frac{1}{2} i\mathbf{k}\cdot(\mathbf{v}+\mathbf{w})\right] F(\mathbf{u}, \mathbf{k}) \tag{14}$$

where

$$F(\mathbf{u}, \mathbf{k}) = \int d\hat{u}'\, u\sigma(\hat{u}\cdot\hat{u}', u) \times\left[\exp\left(-\frac{1}{2} i\hat{k}\cdot\hat{u}'ku\right) - \exp\left(-\frac{1}{2} i\hat{k}\cdot\hat{u}ku\right)\right]$$

It is evident that $F(\mathbf{u}, \mathbf{k})$ is invariant when we interchange the directions of $\mathbf{k}$ and $\mathbf{u}$ but keep their magnitudes unchanged. Therefore

$$F(\mathbf{u}, \mathbf{k}) = \int d\hat{u}'\, u\sigma(\hat{k}\cdot\hat{u}', u) \times\left[\exp\left(-\frac{1}{2} i\hat{u}\cdot\hat{u}'ku\right) - \exp\left(-\frac{1}{2} i\hat{k}\cdot\hat{u}ku\right)\right]$$

Substituting the above expression into Eq. (14), we obtain

$$J(\varphi, \varphi_1) = \int d\hat{u}'\, d\mathbf{k}'\, g(\hat{k}\cdot\hat{u}', k') \times\left\{\varphi_1\left[\frac{k}{2}(\hat{k}+\hat{u}') - \mathbf{k}'\right] \varphi\left[\frac{k}{2}(\hat{k}-\hat{u}') + \mathbf{k}'\right] - \varphi_1(\mathbf{k}-\mathbf{k}')\, \varphi(\mathbf{k}')\right]\right\} \tag{15}$$

where

$$g(\hat{k}\cdot\hat{u}', k') = (2\pi)^{-3} \int d\mathbf{u}\, [\exp(-i\mathbf{u}\cdot\mathbf{k}')\, u\sigma(\hat{k}\cdot\hat{u}', u)$$

For Maxwell molecules, it is easy to obtain from the above formula and Eq. (5) that

$$g(\hat{k}\cdot\hat{u}', k') = g_{\mathrm{M}}(\hat{k}\cdot\hat{u}')\, \delta(\mathbf{k}')$$

so Eq. (15) is reduced to

$$J(\varphi, \varphi_1) = J_{\mathrm{M}}(\varphi, \varphi_1) \equiv \int d\hat{u}'\, g_{\mathrm{M}}(\hat{k}\cdot\hat{u}') \times\left\{\varphi_1\left[\frac{k}{2}(\hat{k}+\hat{u}')\right] \varphi\left[\frac{k}{2}(\hat{k}-\hat{u}')\right] - \varphi_1(\mathbf{k})\, \varphi(0)\right\} \tag{16}$$

In planar geometry, we can write

$$f = f(x, v, \nu, t) \tag{17}$$

where $\nu = \hat{v} \cdot \hat{e}_x = \cos\alpha$; here α is the angle between $\mathbf{v}$ and the x axis. Correspondingly, the generating function is written

$$\varphi = \varphi(x, k, \mu, t) \tag{18}$$

where $\mu = \hat{k} \cdot \hat{e}_x = \cos\theta'$; here θ' is the angle between $\mathbf{k}$ and the x axis. In this case, Eq. (12) may be written as

$$\frac{\partial \varphi}{\partial t} + i\mu \frac{\partial^2 \varphi}{\partial k\, \partial x} + \frac{i(1-\mu^2)}{k} \frac{\partial^2 \varphi}{\partial \mu\, \partial x} = \frac{1}{\varepsilon} J(\varphi, \varphi) \tag{19}$$

In case there is no exchange of energy between the system and the surroundings, the stable stationary state is the Maxwell distribution, i.e.,

$$f_e = (2\pi)^{-3/2} e^{-v^2/2} \tag{20}$$

or, equivalently,

$$\varphi_e = e^{-k^2/2} \tag{21}$$

We consider the Maxwell molecules first. For Maxwell molecules, Eq. (19) becomes

$$\frac{\partial \varphi}{\partial t} - i\mu \frac{\partial^2 \varphi}{\partial k\, \partial x} + \frac{i(1-\mu^2)}{k} \frac{\partial^2 \varphi}{\partial \mu\, \partial x} = \frac{1}{\varepsilon} J_{\mathrm{M}}(\varphi, \varphi) \tag{22}$$

Let the normal solution of Eq. (22) be written in the form

$$\varphi_n = \varphi_0[1 + \xi(x, k, \mu, t)] \tag{23}$$

where

$$\varphi_0 = \rho(x, t) \exp\left[-\frac{1}{2} k^2 \theta(x, t) - ik\mu c(x, t) \right] \tag{24}$$

and $\theta(x, t)$ and $c(x, t)$ are, to zeroth order in ε, the temperature and average velocity field, respectively. Substituting Eqs. (23) and (24) into Eq. (22), we obtain

$$I_{\mathrm{M}}(\xi) = -J_{\mathrm{M}}(\xi, \xi) + \frac{\varepsilon}{\rho} \left[\frac{\partial \xi}{\partial t} + D_0 + D_0 \xi + D_1(\xi) + D_2(\xi) \right] \tag{25}$$

where

$$I_{\mathrm{M}}(\xi)=J_{\mathrm{M}}(1,\xi)+J_{\mathrm{M}}(\xi,1) \tag{26}$$

$$\begin{aligned} D_0=&\left\{\frac{1}{\rho}\frac{\partial\rho}{\partial t}+\frac{1}{\rho}\frac{\partial(\rho c)}{\partial x}\right\}+\left\{\frac{\partial c}{\partial t}+c\frac{\partial c}{\partial x}+\frac{1}{\rho}\frac{\partial(\rho\theta)}{\partial x}\right\}(-ik\mu)\\ &+\left\{\frac{\partial\theta}{\partial t}+c\frac{\partial\theta}{\partial x}+\frac{2}{3}\theta\frac{\partial c}{\partial x}\right\}\left(-\frac{k^2}{2}\right)\\ &+\frac{4}{3}\theta\frac{\partial c}{\partial x}\left\{-\frac{k_1^2}{2}\rho_2(\mu)\right\}+3\theta\frac{\partial\theta}{\partial x}\left(\frac{ik^3}{6}\mu\right) \end{aligned} \tag{27}$$

$$D_1(\xi)=(c-\theta ik\mu)\frac{\partial\xi}{\partial x}+i\mu\frac{\partial^2\xi}{\partial k\,\partial x}+\frac{i(1-\mu^2)}{k}\frac{\partial^2\xi}{\partial\mu\,\partial x} \tag{28}$$

$$D_2(\xi)=\left(\frac{1}{\rho}\frac{\partial\rho}{\partial x}-\frac{k^2}{2}\frac{\partial\theta}{\partial x}-ik\mu\frac{\partial c}{\partial x}\right)\left[i\mu\frac{\partial\xi}{\partial k}+\frac{i(1-\mu^2)}{k}\frac{\partial\xi}{\partial\mu}\right] \tag{29}$$

Denoting

$$e_{nl}=\frac{(-ik)^n}{n!}P_l(\mu),\qquad n=0,1,\ldots;\quad l=n,n-2,\ldots,1\text{ or }0 \tag{30}$$

where $P_l(\mu)$ are Legendre polynomials, we have[10,11]

$$I_{\mathrm{M}}(e_{nl})=-\lambda_{nl}e_{nl} \tag{31}$$

$$\begin{aligned}\lambda_{nl}=2\pi\int_{-1}^{1}dv\,g_{\mathrm{M}}(v)\left[1+\delta_{n,0}\delta_{l,0}-\left(\frac{1+v}{2}\right)^{n/2}P_l\left(\left(\frac{1+v}{2}\right)^{1/2}\right)\right.\\ \left.-\left(\frac{1-v}{2}\right)^{n/2}P_l\left(\left(\frac{1-v}{2}\right)^{1/2}\right)\right]\end{aligned} \tag{32}$$

It should be noticed that

$$\begin{aligned}&\lambda_{00}=\lambda_{11}=\lambda_{20}=0\\ &\lambda_{n,0}=\lambda_{n-1,1},\qquad n=4,6,\ldots\end{aligned} \tag{33}$$

the first three of which are the reflection of the conservation laws of mass, momentum, and energy in the collision. For Maxwell molecules we may choose

$$\lambda_{40}=1 \tag{34}$$

which constitutes the third condition on the units of length, time, and mass alluded to in Section 1.

Let

$$\xi = \sum_{j=0}^{\infty} \varepsilon^j \xi^{(j)} = \sum_{j=0}^{\infty} \varepsilon^j \sum_{n,l} a_{nl}^{(j)}(x, t) e_{nl} \tag{35}$$

where the summation includes $n = 0, 1, \ldots;\ l = n,\ n-2, \ldots, 1$ or 0. Since $\varphi(x, k, \mu, t) = \varphi(x, -k, -\mu, t)$, $(n+l)$ have to be kept even. We may put

$$a_{00}^{(0)} = a_{11}^{(0)} = a_{20}^{(0)} = 0 \tag{36}$$

without loss of generality. Substituting Eq. (35) into Eq. (25), we obtain from the ε^0-order approximation that

$$I_{\mathrm{M}}(\xi^{(0)}) = -J_{\mathrm{M}}(\xi^{(0)}, \xi^{(0)})$$

which has the solution

$$\xi^{(0)} = 0 \tag{37}$$

This solution is unique when the Eqs. (36) are satisfied.

It should be noticed that when we expand the Fourier transform of the distribution function as

$$\varphi = \rho \exp\left(-\frac{1}{2} k^2 \theta - ik\mu c\right)\left(1 + \sum_{n,l} a_{nl} e_{nl}\right)$$

the macroscopic variables can be represented in terms of the coefficients a_{nl}. For example:

Density

$$n = \rho(1 + a_{00})$$

Local velocity

$$\zeta = \left(c + \frac{a_{11}}{1 + a_{00}}\right) \hat{e}_x$$

Temperature

$$T = \theta + \frac{a_{20}}{1 + a_{00}} - \frac{1}{3} \frac{a_{11}^2}{(1 + a_{00})^2}$$

Pressure tensor

$$\mathsf{p} = \rho \left\{\left(\frac{3}{2} a_{22} - \frac{a_{11}^2}{1 + a_{00}}\right) \hat{e}_x \hat{e}_x + \left[\theta(1 + a_{00}) + a_{20} - \frac{1}{2} a_{22}\right] \mathsf{I}\right\}$$

Heat flux vecter

$$\mathbf{q}=\rho\left[\frac{5}{6}a_{31}+\frac{a_{11}^3}{(1+a_{00})^2}-\frac{5}{2}\frac{a_{11}a_{20}}{1+a_{00}}-\frac{a_{11}a_{22}}{1+a_{00}}\right]\hat{e}_x$$

Although all the coefficients $a_{nl}^{(j)}$ do not have a direct physical meaning, use of these coefficients is convenient in the discussion of removing secular terms. They simplify considerably the calculations and the formulas, espcially those in higher order approximations.

3. THE FIRST-ORDER APPROXIMATION

Assume

$$\frac{\partial\rho}{\partial t}=\sum_{i=0}^{\infty}\varepsilon^i p_i,\qquad \frac{\partial c}{\partial t}=\sum_{i=0}^{\infty}\varepsilon^i q_i,\qquad \frac{\partial\theta}{\partial t}=\sum_{i=0}^{\infty}\varepsilon^i s_i \tag{38}$$

where p_i, q_i, and s_i are dependent on ρ, c, θ, and their space derivatives. Hence, Eqs. (38) must be closed equations. It is seen from Eqs. (35), (37), and (38) that the ε-order approximation to Eq. (25) is

$$\begin{aligned}I_{\mathrm{M}}(\xi^{(1)})=\frac{1}{\rho}\Bigg[&\left(\frac{p_0}{\rho}+\frac{1}{\rho}\frac{\partial(\rho\theta)}{\partial x}\right)+e_{11}\left(q_0+c\frac{\partial c}{\partial x}+\frac{1}{\rho}\frac{\partial(\rho\theta)}{\partial x}\right)\\&+e_{20}\left(s_0+c\frac{\partial\theta}{\partial x}+\frac{2}{3}\theta\frac{\partial c}{\partial x}\right)+e_{22}\frac{4}{3}\theta\frac{\partial c}{\partial x}+e_{31}3\theta\frac{\partial\theta}{\partial x}\Bigg]\end{aligned}\tag{39}$$

Equating the coefficients of e_{nl} on both sides of the above equation, we obtain

$$p_0=-\frac{\partial(\rho c)}{\partial x},\qquad q_0=-c\frac{\partial c}{\partial x}-\frac{1}{\rho}\frac{\partial(\rho\theta)}{\partial x},\qquad s_0=-c\frac{\partial\theta}{\partial x}-\frac{2}{3}\theta\frac{\partial c}{\partial x}\tag{40}$$

$$\xi^{(1)}=a_{00}^{(1)}e_{00}+a_{11}^{(1)}e_{11}+a_{20}^{(1)}e_{20}-\frac{4\theta}{3\lambda_{22}\rho}\frac{\partial c}{\partial x}e_{22}-\frac{3\theta}{\lambda_{31}\rho}\frac{\partial\theta}{\partial x}e_{31}\tag{41}$$

where $a_{00}^{(1)}$, $a_{11}^{(1)}$, and $a_{20}^{(1)}$ are arbitrary functions of x and t. These three coefficients were taken to be zero in Cercignani's expansion, but they are kept to be indefinite here and will be defined later.

4. THE SECOND-ORDER APPROXIMATION

Assume that

$$\frac{\partial a_{nl}^{(j)}}{\partial t}=\sum_{i=0}^{\infty}\varepsilon^i\alpha_{nl}^{(j,i)},\qquad (n,l)=(0,0),(1,1),(2,0)\tag{42}$$

where $\alpha_{nl}^{(j,i)}$ are dependent on ρ, c, θ, $a_{00}^{(1)}$, $a_{11}^{(1)}$, $a_{20}^{(1)}$,..., $a_{00}^{(j)}$, $a_{11}^{(j)}$, $a_{20}^{(j)}$, and their space derivatives, so Eqs. (42) must be closed equations also. Differentiating Eq. (41) with respect to t and making use of Eqs. (38) and (42), we get

$$\frac{\partial \xi^{(1)}}{\partial t} = \sum_{i=0}^{\infty} \varepsilon^i \left(\frac{\partial \xi^{(1)}}{\partial t}\right)_i \tag{43}$$

where

$$\begin{aligned}\left(\frac{\partial \xi^{(1)}}{\partial t}\right)_0 &= \alpha_{00}^{(1,0)} e_{00} + \alpha_{11}^{(1,0)} e_{11} + \alpha_{20}^{(1,0)} e_{20} \\ &\quad - \frac{4}{3\lambda_{22}} e_{22} \left(\frac{1}{\rho}\frac{\partial c}{\partial x} s_0 + \frac{\theta}{\rho}\frac{\partial q_0}{\partial x} - \frac{\theta}{\rho^2}\frac{\partial c}{\partial x} p_0\right) \\ &\quad - \frac{3}{\lambda_{31}} e_{31} \left(\frac{1}{\rho}\frac{\partial \theta}{\partial x} s_0 + \frac{\theta}{\rho}\frac{\partial s_0}{\partial x} - \frac{\theta}{\rho^2}\frac{\partial \theta}{\partial x} p_0\right)\end{aligned} \tag{44}$$

Using Eqs. (38), (40), and (30), we obtain from Eq. (27) that

$$D_0 = \sum_{i=0}^{\infty} \varepsilon^i D_{0i} \tag{45}$$

$$D_{00} = \frac{4}{3}\theta \frac{\partial c}{\partial x} e_{22} + 3\theta \frac{\partial \theta}{\partial x} e_{31} \tag{46}$$

$$D_{0i} = \frac{1}{\rho} p_i + e_{11} q_i + e_{20} s_i, \qquad i = 1, 2,... \tag{47}$$

Hence the ε^2-order approximation to Eq. (25) is

$$\begin{aligned}I_{\mathrm{M}}(\xi^{(2)}) = -J_{\mathrm{M}}(\xi^{(1)}, \xi^{(1)}) + \frac{1}{\rho}\Big[D_{00}\xi^{(1)} + D_{01} + D_1(\xi^{(1)}) \\ + D_2(\xi^{(1)}) + \left(\frac{\partial \xi^{(1)}}{\partial t}\right)_0 \Big]\end{aligned} \tag{48}$$

Equating the coefficients of e_{00}, e_{11}, and e_{20} on both sides of the above equation, we get

$$\begin{aligned}&\frac{p_1}{\rho} + \alpha_{00}^{(1,0)} + c\frac{\partial a_{00}^{(1)}}{\partial x} + \frac{1}{\rho}\frac{\partial}{\partial x}(\rho a_{11}^{(1)}) = 0 \\ &q_1 + \alpha_{11}^{(1,0)} + c\frac{\partial a_{11}^{(1)}}{\partial x} + \theta\frac{\partial a_{00}^{(1)}}{\partial x} + \frac{1}{\rho}\frac{\partial}{\partial x}(\rho a_{20}^{(1)}) + a_{11}^{(1)}\frac{\partial c}{\partial x} + \frac{1}{\rho}\frac{\partial}{\partial x}(\rho a_{22}^{(1)}) = 0 \\ &s_1 + \alpha_{20}^{(1,0)} + c\frac{\partial a_{20}^{(1)}}{\partial x} + a_{11}^{(1)}\frac{\partial \theta}{\partial x} + \frac{2}{3}\theta\frac{\partial a_{11}^{(1)}}{\partial x} + \frac{2}{3}a_{20}^{(1)}\frac{\partial c}{\partial x} \\ &\quad + \frac{2}{3}a_{22}^{(1)}\frac{\partial c}{\partial x} + \frac{5}{9\rho}\frac{\partial}{\partial x}(\rho a_{31}^{(1)}) = 0\end{aligned} \tag{49}$$

with $a_{22}^{(1)}$ and $a_{31}^{(1)}$ given by Eq. (41). Six quantities, p_1, q_1, s_1, $\alpha_{00}^{(1,0)}$, $\alpha_{11}^{(1,0)}$, and $\alpha_{20}^{(1,0)}$, are not determined in three equations of Eqs. (49). They may be chosen so that secular terms are removed.

If one puts $N=0$ in Eq. (10), it follows from Eq. (49) that

$$p_1 = q_1 = s_1 = 0 \tag{50}$$

which is equivalent to the Hilbert expansion, and secular terms will appear. In fact, in this case, Eqs. (38) with error $O(\varepsilon^2)$ will become the Euler equations,

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial(\rho c)}{\partial x} &= 0 \\ \frac{\partial c}{\partial t} + c\frac{\partial c}{\partial x} + \frac{1}{\rho}\frac{\partial(\rho\theta)}{\partial x} &= 0 \\ \frac{\partial \theta}{\partial t} + c\frac{\partial \theta}{\partial x} + \frac{2}{3}\theta\frac{\partial c}{\partial x} &= 0 \end{aligned} \tag{51}$$

Now we consider a small disturbance near the equilibrium state:

$$\rho = 1 + \rho', \qquad \theta = 1 + \theta', \qquad c = c' \tag{52}$$

where ρ', θ', and c' are small quantities. In the lowest approximation, Eqs. (51) may be replaced by

$$\frac{\partial \rho'}{\partial t} + \frac{\partial c'}{\partial x} = 0, \qquad \frac{\partial c'}{\partial t} + \frac{\partial}{\partial x}(\rho' + \theta') = 0, \qquad 3\frac{\partial \theta'}{\partial t} + 2\frac{\partial c'}{\partial x} = 0 \tag{53}$$

Assuming a small vibration

$$\rho' = \rho_1 e^{\lambda t + i\kappa x}, \qquad \theta' = \theta_1 e^{\lambda t + i\kappa x}, \qquad c' = c_1 e^{\lambda t + i\kappa x} \tag{54}$$

with real κ, and substituting Eqs. (54) into (53), we get a set of linear algebraic equations in ρ_1, c_1, and θ_1,

$$\begin{pmatrix} \lambda & i\kappa & 0 \\ i\kappa & \lambda & i\kappa \\ 0 & 2i\kappa & 3\lambda \end{pmatrix} \begin{pmatrix} \rho_1 \\ c_1 \\ \theta_1 \end{pmatrix} = 0 \tag{55}$$

Equations (55) have a nontrivial solution under the condition

$$Q_{\kappa^2}(\lambda) \equiv \lambda(3\lambda^2 + 5\kappa^2) = 0 \tag{56}$$

which means Re $\lambda = 0$. Similarly, we can get the equations for $a_{00}^{(1)}$, $a_{11}^{(1)}$, and $a_{20}^{(1)}$ from Eqs. (42), (49), and (50). Making substitutions similar to Eqs. (52), we get linearized equations that are similar to Eqs. (53) but with nonhomogeneous terms, i.e.,

$$\begin{aligned} &\frac{\partial a_{00}^{(1)}}{\partial t} + \frac{\partial a_{11}^{(1)}}{\partial x} = 0 \\ &\frac{\partial a_{00}^{(1)}}{\partial x} + \frac{\partial a_{11}^{(1)}}{\partial t} + \frac{\partial a_{20}^{(1)}}{\partial x} = \frac{4}{3\lambda_{22}} \frac{\partial^2 c'}{\partial x^2} \\ &2\,\frac{\partial a_{11}^{(1)}}{\partial x} + 3\,\frac{\partial a_{20}^{(1)}}{\partial t} = \frac{5}{\lambda_{31}} \frac{\partial^2 \theta'}{\partial x^2} \end{aligned} \tag{57}$$

Hence, for the small vibration (54), the solutions of Eqs. (57) may be expected to be

$$\begin{aligned} a_{00}^{(1)} &= (a_1 t + b_1) e^{\lambda t + i\kappa x} \\ a_{11}^{(1)} &= (a_2 t + b_2) e^{\lambda t + i\kappa x} \\ a_{20}^{(1)} &= (a_3 t + b_3) e^{\lambda t + i\kappa x} \end{aligned} \tag{58}$$

Substituting Eqs. (58) into (57), we get

$$\begin{pmatrix} \lambda & i\kappa & 0 \\ i\kappa & \lambda & i\kappa \\ 0 & 2i\kappa & 3\lambda \end{pmatrix} \begin{pmatrix} a_1 t + b_1 \\ a_2 t + b_2 \\ a_3 t + b_3 \end{pmatrix} = \begin{pmatrix} -a_1 \\ -(4/3\lambda_{22})\kappa^2 c_1 - a_2 \\ -(5/\lambda_{31})\,\kappa^2 \theta_1 - a_3 \end{pmatrix}$$

which may be reduced as follows:

$$\begin{pmatrix} \lambda & i\kappa & 0 \\ i\kappa & \lambda & i\kappa \\ 0 & i\kappa & 3\lambda \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0 \tag{59}$$

$$\begin{pmatrix} \lambda & i\kappa & 0 \\ i\kappa & \lambda & i\kappa \\ 0 & 2i\kappa & 3\lambda \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = -\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} 0 \\ -(4//3\lambda_{22})\,\kappa^2 c_1 \\ -(5/\lambda_{31})\,\kappa^2 \theta_1 \end{pmatrix} \tag{60}$$

In the case $\kappa \neq 0$, we have from Eqs. (59) and (60) that

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \alpha \begin{pmatrix} \rho_1 \\ c_1 \\ \theta_1 \end{pmatrix}$$

where α must be determined so that the right-hand side of Eqs. (60) is orthogonal with the null space of the coefficient matrix. In general, α is not equal to zero, so we get from Eq. (58) that

$$a_{nl}^{(1)} \sim te^{\lambda t + i\kappa x}, \qquad (n, l) = (0, 0), (1, 1), (2, 0) \tag{61}$$

It is clear that they are secular terms, since Re $\lambda = 0$. Hence the choice (50) should be avoided.

However, if we take $N = 1$ in Eq. (10), which, by Eq. (40), amounts to

$$p_1 = 0, \qquad q_1 = -\frac{1}{\rho}\frac{\partial}{\partial x}(\rho a_{22}^{(1)}), \qquad s_1 = -\frac{2}{3}a_{22}^{(1)}\frac{\partial c}{\partial x} - \frac{5}{9\rho}\frac{\partial}{\partial x}(\rho a_{31}^{(1)}) \tag{62}$$

secular terms are removed. In fact, with Eqs. (62), Eqs. (38) become the Navier–Stokes equations to an accuracy $O(\varepsilon^2)$,

$$\begin{aligned}
\frac{\partial \rho}{\partial t} &= -\frac{\partial(\rho c)}{\partial x} \\
\frac{\partial c}{\partial t} &= -c\frac{\partial c}{\partial x} - \frac{1}{\rho}\frac{\partial(\rho\theta)}{\partial x} + \frac{4\varepsilon}{3\lambda_{22}\rho}\frac{\partial}{\partial x}\left(\theta\frac{\partial c}{\partial x}\right) \\
\frac{\partial \theta}{\partial t} &= -c\frac{\partial \theta}{\partial x} - \frac{2}{3}\theta\frac{\partial c}{\partial x} + \frac{8\varepsilon\theta}{9\lambda_{22}\rho}\left(\frac{\partial c}{\partial x}\right)^2 + \frac{5\varepsilon}{3\lambda_{31}\rho}\frac{\partial}{\partial x}\left(\theta\frac{\partial \theta}{\partial x}\right)
\end{aligned} \tag{63}$$

The substitutions (52) and (54) lead to a set of linear algebraic equations for ρ_1, c_1, and θ_1, which has a nontrivial solution under the condition

$$Q_{\kappa^2}(\lambda) = \lambda^3 + \lambda^2(w_1 + w_2)\kappa^2 + \lambda\left(w_1 w_2 \kappa^2 + \frac{5}{3}\right)\kappa^2 + \omega_2\kappa^4 = 0 \tag{64}$$

with $w_1 = 4\varepsilon/(3\lambda_{22}) > 0$ and $w_2 = 5\varepsilon/(3\lambda_{31}) > 0$. It is easy to show that the roots λ of Eq. (64) always have negative real parts for any $\kappa^2 > 0$. In fact:

1. The real root of Eq. (64) must be negative, because all coefficients in this equation are positive.
2. Equation (64) has at least one real root λ_1, which satisfies $-(w_1 + w_2)\kappa^2 < \lambda_1 < 0$, because for any $\lambda_1 \leqslant -(w_1 + w_2)\kappa^2$ we have
$$Q_{\kappa^2}(\lambda_1) \leqslant \lambda_1\left(w_1 w_2 \kappa^2 + \frac{5}{3}\right)\kappa^2 + w_2\kappa^4 < 0$$
3. The other two roots of Eq. (64), λ_2 and λ_3, must satisfy
$$\lambda_2 + \lambda_3 = -(w_1 + w_2)\kappa^2 - \lambda_1 < 0$$

If both λ_2 and λ_3 are real, they are negative because of 1; if they are conjugate complex, they have negative real parts.

Therefore, the solution of Eqs. (63) is stable. It is seen from Eqs. (49) and (62) that

$$
\begin{aligned}
\alpha_{00}^{(1,0)} &= -c\frac{\partial a_{00}^{(1)}}{\partial x} - \frac{1}{\rho}\frac{\partial}{\partial x}(\rho a_{11}^{(1)}) \\
\alpha_{11}^{(1,0)} &= -\frac{\partial}{\partial x}(c a_{11}^{(1)}) - \theta\frac{\partial a_{00}^{(1)}}{\partial x} - \frac{1}{\rho}\frac{\partial}{\partial x}(\rho a_{20}^{(1)}) \\
\alpha_{20}^{(1,0)} &= -c\frac{\partial a_{20}^{(1)}}{\partial x} - \frac{2}{3}\theta\frac{\partial a_{11}^{(1)}}{\partial x} - \frac{2}{3}a_{20}^{(1)}\frac{\partial c}{\partial x} - a_{11}^{(1)}\frac{\partial \theta}{\partial x}
\end{aligned}
\tag{65}
$$

Comparing the coefficients of e_{nl} on both sides of Eqs. (48), we determine all coefficients $a_{nl}^{(2)}$, except $a_{00}^{(2)}$, $a_{11}^{(2)}$, and $a_{20}^{(2)}$, which are arbitrary. The coefficients $a_{22}^{(2)}$ and $a_{31}^{(2)}$ will be used in the next section:

$$
\begin{aligned}
a_{22}^{(2)} = -\frac{1}{\lambda_{22}}\bigg[&\frac{1}{\rho}\left(\frac{\partial a_{22}^{(1)}}{\partial t}\right)_0 + \frac{4}{3\rho}\frac{\partial c}{\partial x}(a_{20}^{(1)} + a_{22}^{(1)}) + \frac{4}{9\rho^2}\frac{\partial}{\partial x}(\rho a_{31}^{(1)}) \\
&+ \frac{4\theta}{3\rho}\frac{\partial a_{11}^{(1)}}{\partial x} + \frac{c}{\rho}\frac{\partial a_{22}^{(1)}}{\partial x} - \lambda_{40}(a_{11}^{(1)})^2\bigg]
\end{aligned}
\tag{66}
$$

$$
\begin{aligned}
a_{31}^{(2)} = -\frac{1}{\lambda_{31}}\bigg[&\frac{1}{\rho}\left(\frac{\partial a_{31}^{(1)}}{\partial t}\right)_0 + \frac{3}{\rho}\frac{\partial \theta}{\partial x}(a_{20}^{(1)} + a_{22}^{(1)}) \\
&+ \frac{11}{5\rho}\frac{\partial c}{\partial x}a_{31}^{(1)} + \frac{6\theta}{5\rho}\frac{\partial a_{22}^{(1)}}{\partial x} + \frac{3\theta}{\rho}\frac{\partial a_{2\text{-}}^{(1)}}{\partial x} \\
&+ \frac{c}{\rho}\frac{\partial a_{31}^{(1)}}{\partial x} - 3\lambda_{40}a_{11}^{(1)}a_{20}^{(1)} + \left(\frac{3}{5}\lambda_{40} - \frac{6}{5}\lambda_{22}\right)a_{11}^{(1)}a_{22}^{(1)}\bigg]
\end{aligned}
\tag{67}
$$

$$
a_{33}^{(2)} = \frac{1}{\lambda_{33}}\left(\frac{4}{5\rho}\frac{\partial\theta}{\partial x}a_{31}^{(1)} + \frac{9\theta}{5\rho}\frac{\partial a_{22}^{(1)}}{\partial x} - \frac{27}{20}\lambda_{40}a_{11}^{(1)}a_{22}^{(1)} - \frac{9}{5}\lambda_{22}a_{11}^{(1)}a_{22}^{(1)}\right)
\tag{68}
$$

$$
\begin{aligned}
a_{40}^{(2)} = -\frac{1}{\lambda_{40}}\bigg[&\frac{10}{3\rho}\frac{\partial\theta}{\partial x}a_{31}^{(1)} + \frac{4\theta}{3\rho}\frac{\partial a_{31}^{(1)}}{\partial x} - (a_{20}^{(1)})^2 h_{\mathrm{M}40}^{20,20} - (a_{22}^{(1)})^2 h_{\mathrm{M}40}^{22,22} \\
&- \frac{6}{5}\lambda_{22}(a_{22}^{(1)})^2 - \left(\frac{4}{3}\lambda_{31} + 2h_{\mathrm{M}40}^{11,31}\right)a_{11}^{(1)}a_{31}^{(1)}\bigg]
\end{aligned}
\tag{69}
$$

$$
\begin{aligned}
a_{42}^{(2)} = -\frac{1}{\lambda_{42}}\bigg[&\frac{8}{3\rho}\frac{\partial\theta}{\partial x}a_{31}^{(1)} + \frac{8\theta}{3\rho}\frac{\partial a_{31}^{(1)}}{\partial x} - \left(2h_{\mathrm{M}42}^{11,31} + \frac{8}{3}\lambda_{31}\right)a_{11}^{(1)}a_{31}^{(1)} \\
&- (2h_{\mathrm{M}42}^{20,20} + b\lambda^{22})a_{20}^{(1)}a_{22}^{(1)} - \left(h_{\mathrm{M}42}^{22,22} + \frac{12}{7}\lambda_{22}\right)(a_{22}^{(1)})^2\bigg]
\end{aligned}
\tag{70}
$$

$$a_{44}^{(2)}=\frac{1}{\lambda_{44}}\left(h_{M44}^{22,22}+\frac{108}{35}\lambda_{22}\right)(a_{22}^{(1)})^2 \tag{71}$$

$$a_{51}^{(2)}=\frac{1}{\lambda_{51}}\left[(2h_{M51}^{22,31}+4\lambda_{31}+4\lambda_{22})a_{22}^{(1)}a_{31}^{(1)}+(2h_{M51}^{20,31}+10\lambda_{31})a_{20}^{(1)}a_{31}^{(1)}\right] \tag{72}$$

$$a_{53}^{(2)}=\frac{1}{\lambda_{53}}(2h_{M53}^{22,31}+6\lambda_{31}+6\lambda_{22})a_{22}^{(1)}a_{31}^{(1)} \tag{73}$$

$$a_{60}^{(2)}=\frac{1}{\lambda_{60}}\left(h_{M60}^{31,31}+\frac{20}{3}\lambda_{31}\right)(a_{31}^{(1)})^2 \tag{74}$$

$$a_{62}^{(2)}=\frac{1}{\lambda_{62}}\left(h_{M62}^{31,31}+\frac{40}{3}\lambda_{31}\right)(a_{31}^{(1)})^2 \tag{75}$$

and $a_{nl}^{(2)}=0$ for all other (n, l). The coefficients $h_{Mnl}^{n'l'n''l''}$ in Eqs. (79)–(75) are defined by

$$\frac{1}{2}\left[J_M(e_{n'l'}, e_{n''l''})+J_M(e_{n''l''}, e_{n'l'})\right]=\sum_{nl} h_{Mnl}^{n'l'n''l''}e_{nl} \tag{76}$$

It is not difficult to get

$$h_{M22}^{11,11}=\lambda_{40}, \qquad h_{M31}^{11,20}=\frac{3}{2}\lambda_{40}$$

$$h_{M31}^{11,22}=-\frac{3}{10}\lambda_{40}, \qquad h_{M33}^{11,22}=\frac{27}{40}\lambda_{40},\ldots$$

It may be seen that $h_{Mnl}^{n'l'n''l''}=0$ unless $n=n'+n''$, and that

$$\begin{aligned} &h_{M00}^{n'l'n''l''}=h_{M11}^{n'l'n''l''}=h_{M20}^{n'l'n''l''}=0, \qquad h_{Mnl}^{n'l'n''l''}=h_{Mnl}^{n''l''n'l'} \\ &h_{Mnl}^{n'l'n''l''}=0 \qquad \text{if} \quad |l'-l''|>l \quad \text{or} \quad l>l'+l'' \end{aligned} \tag{77}$$

The symbols $(\partial a_{nl}^{(j)}/\partial t)_0$ in Eqs. (6) and (67) are to be understood similarly to those in Eq. (43).

5. THE THIRD-ORDER APPROXIMATION

The ε^3-order approximation to Eq. (25) is

$$\begin{aligned} I_M(\xi^{(3)})=\frac{1}{\rho}\Bigg[&D_{00}\xi^{(2)}+D_{01}\xi^{(1)}+D_{02}+D_1(\xi^{(2)})+D_2(\xi^{(2)}) \\ &+\left(\frac{\partial\xi^{(1)}}{\partial t}\right)_1+\left(\frac{\partial\xi^{(2)}}{\partial t}\right)_0\Bigg] \\ &-\left[J_M(\xi^{(1)}, \xi^{(2)})+J_M(\xi^{(2)}, \xi^{(1)})\right] \end{aligned} \tag{78}$$

Equating the coefficiens of e_{00}, e_{11}, and e_{20} on both sides of Eq. (78), we get

$$\frac{1}{\rho}p_2+\alpha_{00}^{(1,1)}+\alpha_{00}^{(2,0)}+c\frac{\partial a_{00}^{(2)}}{\partial x}+\frac{1}{\rho}\frac{\partial}{\partial x}(\rho a_{11}^{(2)})=0$$

$$q_2+\alpha_{11}^{(1,1)}+\alpha_{11}^{(2,0)}+a_{00}^{(1)}q_1+c\frac{\partial a_{11}^{(2)}}{\partial x}+\frac{1}{\rho}\frac{\partial}{\partial x}[\rho(a_{20}^{(2)}+a_{22}^{(2)})]$$

$$+\theta\frac{\partial a_{00}^{(2)}}{\partial x}+a_{11}^{(2)}\frac{\partial c}{\partial x}=0 \tag{79}$$

$$s_2+a_{20}^{(1,1)}+a_{20}^{(2,0)}+\frac{2}{3}a_{11}^{(1)}q_1+a_{00}^{(1)}s_1+c\frac{\partial a_{20}^{(2)}}{\partial x}+\frac{2}{3}\theta\frac{\partial a_{11}^{(2)}}{\partial x}+\frac{5}{9\rho}\frac{\partial}{\partial x}(\rho a_{31}^{(2)})$$

$$+\frac{2}{3}\frac{\partial c}{\partial x}(a_{20}^{(2)}+a_{22}^{(2)})+a_{11}^{(2)}\frac{\partial\theta}{\partial x}=0$$

where nine quantities are to be determined. If one puts $N=2$ in Eq. (10), one obtains from Eqs. (79) that

$$a_{nl}^{(1)}=a_{nl}^{(2)}=0,\qquad \alpha_{nl}^{(1,1)}=\alpha_{nl}^{(2,0)}=0,\qquad (n,l)=(0,0),(1,1),(2,0) \tag{80}$$

which is equivalent to the Enskog expansion, and the secular terms appear again. In fact, with Eqs. (80), (66), and (67), Eqs. (38) become

$$\begin{aligned}
&\rho_t+(\rho c)_x=0\\
&\rho(c_t+cc_x)+(\rho\theta)_x+\bar{\sigma}_x=0\\
&\theta_t+c\theta_x+\frac{2}{3}\theta c_x+\frac{5}{9\rho}\bar{\delta}_x+\frac{2}{3\rho}c_x\bar{\sigma}=0\\
&\bar{\sigma}=-\frac{4\omega}{3}\theta c_x+\frac{2\omega^2}{9\rho}\left\{4\theta c_x^2+9(\theta\theta_x)_x-6\theta\left[\frac{1}{\rho}(\rho\theta)_x\right]_x\right\}\\
&\bar{\delta}=-\frac{9}{2}\omega\theta\theta_x+\frac{3\omega^2\theta}{20\rho^2}(95\rho c_x\theta_x-16\theta\rho_x c_x-14\rho\theta c_{xx})
\end{aligned} \tag{81}$$

to an accuracy $O(\varepsilon^3)$. Here subscripts denote differentiation with respect to t or x, and

$$\omega=\frac{\varepsilon}{\lambda_{22}}=\frac{2\varepsilon}{3\lambda_{31}},\qquad \bar{\delta}=\rho\varepsilon a_{31}^{(1)}+\rho\varepsilon^2 a_{31}^{(2)},\qquad \bar{\sigma}=\rho\varepsilon a_{22}^{(1)}+\rho\varepsilon^2 a_{22}^{(2)}$$

According to the procedure outlined by Bobylev,[12] rewriting x/w and t/w as x and t, using Eqs. (52) and (54), we get a set of linear algebraic equations for ρ_1, θ_1, and c_1, which has a nontrivial solution under the condition

$$Q_{\kappa^2}(\lambda) = 18\lambda^3 + 69\lambda^2\kappa^2 + \lambda\kappa^2(30 + 97\kappa^2 - 14\kappa^4) + 15\kappa^4(3 + 4\kappa^2) = 0 \tag{82}$$

Equation (82) has a positive root as long as κ^2 is large enough, because (1) $Q_{\kappa^2}(0) > 0$ and $Q_{\kappa^2}(+\infty) > 0$ for any given $\kappa^2 > 0$; and (2) $Q_{\kappa^2}(\kappa^2) < 0$ for a sufficiently large $k^2 > 0$. Hence, Eqs. (80) lead to secular terms.

If one puts $N = 0$, i.e., $p_2 = q_2 = s_2 = \alpha_{00}^{(1,1)} = \alpha_{11}^{(1,1)} = \alpha_{20}^{(1,1)} = 0$, the secular terms still appear. This is seen by arguments similar to those following Eqs. (50).

We find that the secular terms can be removed by putting $N = 1$, i.e., by requiring

$$p_2 = q_2 = s_2 = 0 \tag{83}$$

$$\begin{aligned} \alpha_{00}^{(1,1)} &= 0 \\ \alpha_{11}^{(1,1)} &= -a_{00}^{(1)} q_1 - \frac{1}{\rho}\frac{\partial}{\partial x}(\rho a_{22}^{(2)}) \\ \alpha_{20}^{(1,1)} &= -\frac{2}{3} a_{11}^{(1)} q_1 - a_{00}^{(1)} s_1 - \frac{2}{3}\frac{\partial c}{\partial x} a_{22}^{(2)} - \frac{5}{9\rho}\frac{\partial}{\partial x}(\rho a_{31}^{(2)}) \end{aligned} \tag{84}$$

and

$$\begin{aligned} \alpha_{00}^{(2,0)} &= -c\frac{\partial a_{00}^{(2)}}{\partial x} - \frac{1}{\rho}\frac{\partial}{\partial x}(\rho a_{11}^{(2)}) \\ \alpha_{11}^{(2,0)} &= -\frac{\partial}{\partial x}(c a_{11}^{(2)}) - \theta\frac{\partial a_{00}^{(2)}}{\partial x} - \frac{1}{\rho}\frac{\partial}{\partial x}(\rho a_{20}^{(2)}) \\ \alpha_{20}^{(2,0)} &= -c\frac{\partial a_{20}^{(2)}}{\partial x} - \frac{2}{3}\theta\frac{\partial a_{11}^{(2)}}{\partial x} - \frac{2}{3}\frac{\partial c}{\partial x} a_{20}^{(2)} - a_{11}^{(2)}\frac{\partial \theta}{\partial x} \end{aligned} \tag{85}$$

In fact, the equations for ρ, c, and θ to an accuracy $O(\varepsilon^3)$ are still Eqs. (63), so the equiliblium values $\rho = 1$, $\theta = 1$, and $c = 0$ should be obtained as $t \to \infty$. The equations for $a_{00}^{(1)}$, $a_{11}^{(1)}$, and $a_{20}^{(1)}$ to an accuracy $O(\varepsilon^2)$ can be written as

$$\begin{aligned} \frac{\partial a_{00}^{(1)}}{\partial t} &= -c\frac{\partial a_{00}^{(1)}}{\partial x} - \frac{1}{\rho}\frac{\partial}{\partial x}(\rho a_{11}^{(1)}) \\ \frac{\partial a_{11}^{(1)}}{\partial t} &= -\frac{\partial}{\partial x}\left(c a_{11}^{(1)}\right) - \theta\frac{\partial a_{00}^{(1)}}{\partial x} - \frac{1}{\rho}\frac{\partial}{\partial x}(\rho a_{20}^{(1)}) - \varepsilon\left[a_{00}^{(1)} q_1 + \frac{1}{\rho}\frac{\partial}{\partial x}(\rho a_{22}^{(2)})\right] \\ \frac{\partial a_{20}^{(1)}}{\partial t} &= -c\frac{\partial a_{20}^{(1)}}{\partial x} - \frac{2}{3}\theta\frac{\partial a_{11}^{(1)}}{\partial x} - \frac{2}{3}\frac{\partial c}{\partial x} a_{20}^{(1)} - a_{11}^{(1)}\frac{\partial \theta}{\partial x} \\ &\quad - \varepsilon\left[\frac{2}{3} a_{11}^{(1)} q_1 + a_{00}^{(1)} s_1 + \frac{5}{9\rho}\frac{\partial}{\partial x}(\rho a_{31}^{(2)}) + \frac{2}{3}\frac{\partial c}{\partial x} a_{22}^{(2)}\right] \end{aligned} \tag{86}$$

It can be shown that the set of Eqs. (86) has a stable solution also. The proof is easy. The values $\rho=1$, $\theta=1$, and $c=0$ are reached after a sufficiently long time, so it is known from Eqs. (41) and (62) that

$$a_{nl}^{(1)}=0, \qquad (n, l)=(2, 2), (3, 1),...; \qquad q_1=s_1=0 \tag{87}$$

and from Eqs. (66) and (67) that

$$\begin{aligned} a_{22}^{(2)} &= \frac{1}{\lambda_{22}}\left[\lambda_{40}(a_{11}^{(1)})^2-\frac{4}{3}\frac{\partial a_{11}^{(1)}}{\partial x}\right] \\ a_{31}^{(2)} &= \frac{1}{\lambda_{31}}\left(3\lambda_{40}a_{11}^{(1)}a_{20}^{(1)}-3\frac{\partial a_{20}^{(1)}}{\partial x}\right) \end{aligned} \tag{88}$$

Hence Eqs. (86) become

$$\begin{aligned} \frac{\partial a_{20}^{(1)}}{\partial t} &= -\frac{\partial a_{11}^{(1)}}{\partial x} \\ \frac{\partial a_{11}^{(1)}}{\partial t} &= -\frac{\partial a_{00}^{(1)}}{\partial x}-\frac{\partial a_{20}^{(1)}}{\partial x}+\frac{\varepsilon}{\lambda_{22}}\frac{\partial}{\partial x}\left[\frac{4}{3}\frac{\partial a_{11}^{(1)}}{\partial x}-\lambda_{40}(a_{11}^{(1)})^2\right] \\ \frac{\partial a_{20}^{(1)}}{\partial t} &= -\frac{2}{3}\frac{\partial a_{11}^{(1)}}{\partial x}+\frac{\varepsilon}{\lambda_{31}}\frac{\partial}{\partial x}\left(\frac{5}{3}\frac{\partial a_{20}^{(1)}}{\partial x}-\frac{5}{3}\lambda_{40}a_{11}^{(1)}a_{20}^{(1)}\right) \end{aligned} \tag{89}$$

which has a steady-state solution $a_{00}^{(1)}=a_{11}^{(1)}=a_{20}^{(1)}=0$. It can be seen from a discussion similar to that relating to Eqs. (63) that this solution is stable.

The requirements (84) and (85) determine the time evolution of ξ (in ε^2 order), which in turn determines the normal solution in k space through Eq. (23). However, the explicit reltion of Eqs. (84) to the usual fluid equations is difficult to obtain, due to the complexity of the formulas relating the expansion coefficients $\alpha_{nl}^{(j,i)}$ to the usual hydrodynamic quantities.

6. EXTENSION TO NON-MAXWELL MOLECULES

Before discussing the higher order approximations, we extend the discussion in the foregoing sections to non-Maxwell molecules.

Substituting Eq. (23) into Eq. (19), we obtain for non-Maxwell molecules

$$I(\xi)=-J'(\xi, \xi)+\frac{\varepsilon}{\rho}\left[\frac{\partial \xi}{\partial t}+D_0+D_0(\xi)+D_1(\xi)+D_2(\xi)\right] \tag{90}$$

where

$$I(\xi)=\int d\hat{u}'\, d\mathbf{k}'\, g(\hat{k}\cdot\hat{u}, k')[\exp(-\theta k'^2)]$$
$$\times\left([\exp(\theta k\mathbf{k}'\cdot\hat{u}')]\left\{\xi\left[\frac{k}{2}(\hat{k}+\hat{u}')-\mathbf{k}'\right]+\xi\left[\frac{k}{2}(\hat{k}-\hat{u}')+\mathbf{k}'\right]\right\}\right.$$
$$\left.-\ [\exp(\theta\mathbf{k}\cdot\mathbf{k}')][\xi(\mathbf{k}-\mathbf{k}')+\xi(\mathbf{k}')]\right) \tag{91}$$

$$J'(\xi,\xi_1)=\int d\hat{u}'\, d\mathbf{k}'\, g(\hat{k}\cdot\hat{u}', k')[\exp(-\theta k'^2)]$$
$$\times\left\{[\exp(\theta k\mathbf{k}'\cdot\hat{u}')]\ \xi_1\left[\frac{k}{2}(\hat{k}+\hat{u}')-\mathbf{k}'\right]\xi\left[\frac{k}{2}(\hat{k}-\hat{u}')+\mathbf{k}'\right]\right.$$
$$\left.-\ [\exp(\theta\mathbf{k}\cdot\mathbf{k}')]\ \xi_1(\mathbf{k}-\mathbf{k}')\ \xi(\mathbf{k}')\right\} \tag{92}$$

Assuming that φ_n and φ_0 are Fourier transforms of

$$f_n=f_0(1+h) \tag{93}$$

$$f_0=f_0(\mathbf{v})=\rho(2\pi\theta)^{-3/2}\exp(-v_1^2/2\theta) \tag{94}$$

respectively, where $\mathbf{v}_1=\mathbf{v}-c\hat{e}_x$, $h=h(\mathbf{v})$, we get

$$\varphi_0\xi=\int[\exp(-i\mathbf{k}\cdot\mathbf{v})]\,f_0h\,d\mathbf{v} \tag{95}$$

Substituting Eq. (93) into Eq. (1), we have

$$C(f_n,f_n)=\rho f_0\tilde{I}(h)+C(f_0h,f_0h) \tag{96}$$

where

$$\tilde{I}(h)=\int d\mathbf{w}\,d\hat{u}'\,u\sigma(\hat{u}\cdot\hat{u}',u)\,\rho^{-1}f_0(\mathbf{w})[h(\mathbf{w}')+h(\mathbf{v}')-h(\mathbf{w})-h(\mathbf{v})] \tag{97}$$

We obtain from Eqs. (91) and (97) that

$$\varphi_0I(\xi)=\int[\exp(-i\mathbf{k}\cdot\mathbf{v})]\,f_0\tilde{I}(h)\,d\mathbf{v} \tag{98}$$

Assume

$$E'_{nl}=\frac{(-1)^{(n-l)/2}(n-l)!!}{n!\,\theta^{(n+l)/2}}\,v_1^l\,P_l(\hat{v}_1\cdot\hat{e}_x)\,L^{l+1/2}_{(n-l)/2}\left(\frac{v_1^2}{2\theta}\right) \tag{99}$$

where L_n^l are generalized Laguerre polynomials. It is easy to show that

$$\varphi_0 e_{nl} = \int [\exp(-i\mathbf{k}\cdot\mathbf{v})] f_0 E'_{nl}\, d\mathbf{v} \tag{100}$$

$$\int f_0 E'_{nl} E'_{n'l'}\, d\mathbf{v} = \rho\theta^{-n} N_{nl}\, \delta_{nn'}\, \delta_{ll'} \tag{101}$$

$$N_{nl} = \frac{1}{2l+1} \frac{(n-l)!!}{(n+l)!!} \frac{(n+l+1)!}{(n!)^2} \tag{102}$$

Suppose that the distribution function (17) can be expanded as

$$f = f_0 \left[1 + \sum_{nl} a_{nl}(x, t)\, E'_{nl}(\mathbf{v}) \right] \tag{103}$$

This means that the integral

$$\int f_0^{-1} f^2\, d\mathbf{v} \tag{104}$$

exists and is bounded. Such functions span a Hilbert space $\mathscr{H}$. It is easy to see from Eq. (100) that Eq. (103) is equivalent to

$$\varphi = \varphi_0 \left[1 + \sum_{nl} a_{nl}(x, t)\, e_{nl} \right] \tag{105}$$

In case φ is the normal solution φ_n, we have $a_{nl} = \sum_{j=1}^{\infty} \varepsilon^j a_{nl}^{(j)}$. If $f \in \mathscr{H}$, we will denote $\varphi \in \mathscr{H}$ also in short. That is, we use the same symbol $\mathscr{H}$ for the space transformed from $\mathscr{H}$ by Eq. (11). Supposing that[11]

$$\tilde{I}(E'_{nl}) = -\sum_{n'l'} \lambda_{nl}^{n'l'}(\theta) E'_{n'l'} \tag{106}$$

and using Eq. (101), we get

$$\begin{aligned} \lambda_{nl}^{n'l'}(\theta) &= -\theta^{n'} \rho^{-1} N_{n'l'}^{-1} \int f_0 \tilde{I}(E'_{nl})\, E'_{n'l'}\, d\mathbf{v} \\ &= \frac{\theta^{n'}}{4\rho^2} N_{n'l'}^{-1} \int d\mathbf{v}\, d\mathbf{w}\, d\hat{u}'\, u\sigma(\hat{u}\cdot\hat{u}', u)\, f_0(\mathbf{v})\, f_0(\mathbf{w}) \\ &\quad \times [E'_{nl}(\mathbf{w}') + E'_{nl}(\mathbf{v}') - E'_{nl}(\mathbf{w}) - E'_{nl}(\mathbf{v})] \\ &\quad \times [E'_{n'l'}(\mathbf{w}') + E'_{n'l'}(\mathbf{v}') - E'_{n'l'}(\mathbf{w}) - E'_{n'l'}(\mathbf{v})] \end{aligned} \tag{107}$$

where Eq. (97) has ben used. It is clear that $\lambda_{nl}^{n'l'}(\theta) > 0$ and $\lambda_{nl}^{n'l'}(\theta)$ depends on θ in general. Denote $\lambda_{nl}^{n'l'}(\theta = 1) \equiv \lambda_{nl}^{n'l'}$. For non-Maxwell molecules, we take

$$\lambda_{40}^{40} = 1 \tag{108}$$

instead of Eq. (34) for determining the units of time and length.

It can be proved from Eqs. (95), (98), (100), and (106) that

$$I(e_{nl}) = -\sum_{n'l'} \lambda_{nl}^{n'l'}(\theta) e_{n'l'} \tag{109}$$

Denoting

$$\tilde{e}_{nl}(\theta) = \theta^{n/2} N_{nl}^{-1/2} e_{nl}$$

we define the inner product

$$(\tilde{e}_{nl}(\theta), \tilde{e}_{n'l'}(\theta)) = \delta_{nn'} \delta_{ll'} \tag{110}$$

and take $\{\tilde{e}_{nl}(\theta)\}$ as a basis for $\mathscr{H}$. It is easy to prove that

$$(\tilde{e}_{n'l'}(\theta), I[\tilde{e}_{nl}(\theta)]) = -\lambda_{nl}^{n'l'}(\theta) \left[\frac{N_{n'l'} \theta^{n}}{N_{nl} \theta^{n'}} \right]^{1/2}$$

Taking into account Eq. (107), we find that I is a real, symmetric operator and that the zero eigenvalue is threefold degenerate with corresponding eigenfunctions e_{00}, e_{11}, e_{20}, i.e.,

$$I(e_{00}) = I(e_{11}) = I(e_{20}) = 0 \tag{111}$$

Denote the subspace spaned by e_{00}, e_{11}, and e_{20} as $\mathscr{H}_1$, and the subspace orthogonal to $\mathscr{H}_1$ as $\mathscr{H}_2$. Since I has a unique inverse in $\mathscr{H}_2$, we have

$$e_{nl} = -\sum_{n'l'} \mu_{nl}^{n'l'}(\theta)\, I(e_{n'l'}), \qquad \left.\begin{matrix}(n', l')\\(n, l)\end{matrix}\right\} = (2, 2), (3, 1),... \tag{112}$$

and it is quite evident that

$$\sum_{n'l'} \lambda_{nl}^{n'l'}(\theta)\, \mu_{n'l'}^{n''l''}(\theta) = \delta_{nn''} \delta_{ll''} \tag{113}$$

We obtain from Eq. (107) that

$$\lambda_{nl}^{n'l'}(\theta) = \lambda_{nl}^{n'l}(\theta) \delta_{ll'}, \qquad \mu_{nl}^{n'l'}(\theta) = \mu_{nl}^{n'l}(\theta) \delta_{ll'}$$

Instead of Eq. (76), we now define $h_{nl}^{n'l'n''l''}(\theta)$ by

$$\frac{1}{2}\left[J'(e_{n'l'}, e_{n''l''}) + J'(e_{n''l''}, e_{n'l'})\right] = \sum_{nl} h_{nl}^{n'l'n''l''}(\theta) e_{nl}$$

and denote

$$h_{nl}^{n'l'n''l''} \equiv h_{nl}^{n'l'n''l''}(\theta = 1)$$

It is easy to see that $h_{nl}^{n'l'n''l''}$ has properties similar to Eq. (77), but it may not be zero even though $n \neq n' + n''$.

For non-Maxwell molecules, Eqs. (40), (62), (65), and (83)–(85) may be kept unaltered. Substituting Eq. (35) into Eq. (90), we obtain from the zeroth approximation that

$$\xi^{(0)} = 0$$

and from the first approximation that

$$\begin{aligned}
a_{n,0}^{(1)} &= 0, \qquad n = 4, 6, \ldots \\
a_{n,1}^{(1)} &= -\frac{3\theta}{\rho}\frac{\partial\theta}{\partial x}\mu_{31}^{n1}(\theta), \qquad n = 3, 5, \ldots \\
a_{n,2}^{(1)} &= -\frac{4\theta}{3\rho}\frac{\partial c}{\partial x}\mu_{22}^{n,2}(\theta), \qquad n = 2, 4, \ldots \\
a_{nl}^{(1)} &= 0, \qquad l = 3, 4, \ldots
\end{aligned} \tag{114}$$

Therefore Eq. (63) should be rewritten as

$$\begin{aligned}
\frac{\partial\rho}{\partial t} &= -\frac{\partial(\rho c)}{\partial x} \\
\frac{\partial c}{\partial t} &= -c\frac{\partial c}{\partial x} - \frac{1}{\rho}\frac{\partial(\rho\theta)}{\partial x} + \frac{4\varepsilon}{3\rho}\frac{\partial}{\partial x}\left[\mu_{22}^{22}(\theta)\theta\frac{\partial c}{\partial x}\right] \\
\frac{\partial\theta}{\partial t} &= -c\frac{\partial\theta}{\partial x} - \frac{2}{3}\theta\frac{\partial c}{\partial x} + \frac{8\varepsilon\theta}{9\rho}\mu_{22}^{22}(\theta)\left(\frac{\partial c}{\partial x}\right)^2 \\
&\quad + \frac{5\varepsilon}{3\rho}\frac{\partial}{\partial x}\left[\mu_{31}^{31}(\theta)\theta\frac{\partial\theta}{\partial x}\right]
\end{aligned} \tag{115}$$

The solution of Eqs. (115) is stable, so the value $\rho = 1$, $\theta = 1$, and $c = 0$ are reached after a sufficiently long time. Therefore we get from Eq. (114) that

$$a_{nl}^{(1)} = 0, \qquad (n, l) = (2, 2), (3, 1), \ldots \tag{116}$$

In this case, the ε^2-order approximation to Eq. (90) is

$$I(\xi^{(2)}) = \frac{4}{3}\frac{\partial a_{11}^{(1)}}{\partial x} e_{22} + 3\frac{\partial a_{20}^{(1)}}{\partial x} e_{31} - \sum_{nl}\sum_{n'l'n''l''} a_{n'l'}^{(1)} a_{n''l''}^{(1)} h_{nl}^{n'l'n''l''} e_{nl}$$

$$\left.\begin{matrix}(n', l')\\(n'', l'')\end{matrix}\right\} = (0, 0), (1, 1), (2, 0) \tag{117}$$

Hence

$$\begin{aligned}
a_{n,0}^{(2)} &= \sum_{n^*} \mu_{n^*0}^{n0} \sum_{n'l'n''l''} a_{n'l'}^{(1)} a_{n''l''}^{(1)} h_{n^*0}^{n'l'n''l''}, \qquad n = 4, 6, \ldots\\
a_{n,1}^{(2)} &= -3\frac{\partial a_{20}^{(1)}}{\partial x}\mu_{31}^{n1} + \sum_{n^*}\mu_{n^*1}^{n1}\sum_{n'l'n''l''} a_{n'l'}^{(1)} a_{n''l''}^{(1)} h_{n^*1}^{n'l'n''l''}, \qquad n = 3, 5, \ldots\\
a_{n,2}^{(2)} &= -\frac{4}{3}\frac{\partial a_{11}^{(1)}}{\partial x}\mu_{22}^{n,2} + \sum_{n^*}\mu_{n^*2}^{n2}\sum_{n'l'n''l''} a_{n'l'}^{(1)} a_{n''l''}^{(1)} h_{n^*2}^{n'l'n''l''}, \qquad n = 2, 4, \ldots\\
a_{n,l}^{(2)} &= 0, \qquad l = 3, 4, \ldots
\end{aligned} \tag{118}$$

We find that the form of Eqs. (86) is not changed, but $a_{nl}^{(2)}$ has a different expression from Eq. (66). It is known from Eq. (118) that Eqs. (86) will be reduced to

$$\begin{aligned}
\frac{\partial a_{00}^{(1)}}{\partial t} &= -\frac{\partial a_{11}^{(1)}}{\partial x}\\
\frac{\partial a_{11}^{(1)}}{\partial t} &= -\frac{\partial a_{00}^{(1)}}{\partial x} - \frac{\partial a_{20}^{(1)}}{\partial x} + \frac{4\varepsilon}{3}\mu_{22}^{22}\frac{\partial^2 a_{11}^{(1)}}{\partial x^2} - 2\varepsilon a_{11}^{(1)}\frac{\partial a_{11}^{(1)}}{\partial x}\sum_{n^*}\mu_{n^*2}^{n2} h_{n^*2}^{11,11}\\
\frac{\partial a_{20}^{(1)}}{\partial t} &= -\frac{2}{3}\frac{\partial a_{11}^{(1)}}{\partial x} + \frac{5\varepsilon}{3}\mu_{31}^{31}\frac{\partial^2 a_{20}^{(1)}}{\partial x} - \frac{10\varepsilon}{4}\frac{\partial}{\partial x}(a_{11}^{(1)} a_{20}^{(1)})\sum_{n^*}\mu_{n^*1}^{31} h_{n^*1}^{20,11}
\end{aligned} \tag{119}$$

after a sufficiently long time. The solution of Eqs. (119) is stable, too, so the values $a_{00}^{(1)} = a_{11}^{(1)} = a_{20}^{(1)} = 0$ will finally be reached. Combining this conclusion with Eq. (116), we get that

$$a_{nl}^{(1)} = 0, \qquad (n, l) = (0, 0), (1, 1), \ldots \tag{120}$$

and from Eq. (118) that

$$a_{nl}^{(2)} = 0, \qquad (n, l) = (2, 2), (3, 1), \ldots \tag{121}$$

In this case we find that the ε^3-order approximation to Eq. (90) is

$$I(\xi^{(3)}) = \frac{4}{3}\frac{\partial a_{11}^{(2)}}{\partial x} e_{22} + 3\frac{\partial a_{20}^{(2)}}{\partial x} e_{31} \tag{122}$$

so we have that

$$
\begin{aligned}
&a_{n,0}^{(3)}=0, \qquad n=4,6,\ldots \\
&a_{n,1}^{(3)}=-3\frac{\partial a_{20}^{(2)}}{\partial x}\mu_{31}^{n,1}, \qquad n=3,5,\ldots \\
&a_{n,2}^{(3)}=-\frac{4}{3}\frac{\partial a_{11}^{(2)}}{\partial x}\mu_{22}^{n,2}, \qquad n=2,4,\ldots \\
&a_{n,l}^{(3)}=0, \qquad l=3,4,\ldots
\end{aligned}
\tag{123}
$$

It is quite evident from the above discussion that Eqs. (40), (62), (65), and (3)–(85) do not cause secular terms, neither in the case of Maxwell molecules, nor in that of non-Maxwell molecules.

7. THE HIGHER ORDER APPROXIMATIONS

It can be proved by induction that the expansion without secular terms can be obtained for $j\geqslant 3$ by putting $N=1$ in Eq. (10), which means that

$$
p_j=q_j=s_j=0, \qquad \alpha_{nl}^{(j-i,i)}=0, \qquad i=2,3,\ldots,j \tag{124}
$$

$$
\begin{aligned}
&\alpha_{00}^{(j-1,1)}=0 \\
&\alpha_{11}^{(j-1,1)}=-a_{00}^{(j-1)}q_1-\frac{1}{\rho}\frac{\partial}{\partial x}(\rho a_{22}^{(j)}) \\
&\alpha_{20}^{(j-1,1)}=-\frac{2}{3}a_{11}^{(j-1)}q_1-a_{00}^{(j-1)}s_1-\frac{2}{3}\frac{\partial c}{\partial x}a_{22}^{(j)}-\frac{5}{9\rho}\frac{\partial}{\partial x}(\rho a_{31}^{(j)})
\end{aligned}
\tag{125}
$$

$$
\begin{aligned}
&\alpha_{00}^{(j,0)}=-c\frac{\partial a_{00}^{(j)}}{\partial x}-\frac{1}{\rho}\frac{\partial}{\partial x}(\rho a_{11}^{(j)}) \\
&\alpha_{11}^{(j,0)}=-\frac{\partial}{\partial x}(ca_{11}^{(j)})-\theta\frac{\partial a_{00}^{(j)}}{\partial x}-\frac{1}{\rho}\frac{\partial}{\partial x}(\rho a_{20}^{(j)}) \\
&\alpha_{20}^{(j,0)}=-c\frac{\partial a_{20}^{(j)}}{\partial x}-\frac{2}{3}\theta\frac{\partial a_{11}^{(j)}}{\partial x}-\frac{2}{3}\frac{\partial c}{\partial x}a_{20}^{(j)}-\frac{\partial\theta}{\partial x}a_{11}^{(j)}
\end{aligned}
\tag{126}
$$

This conclusion is correct for non-Maxwell molecules as well as Maxwell molecules. The proof is not difficult.

In fact, it is seen from Eqs. (124)–(126) that Eqs. (115) with the special form (63), together with Eqs. (86), become exact equations; the exact equations for $a_{00}^{(j-1)}$, $a_{11}^{(j-1)}$, and $a_{20}^{(j-1)}$ are written as

$$
\begin{aligned}
\frac{\partial a_{00}^{(j-1)}}{\partial t} &= -c\,\frac{\partial a_{00}^{(j-1)}}{\partial x} - \frac{1}{\rho}\frac{\partial}{\partial x}(\rho a_{11}^{(j-1)}) \\
\frac{\partial a_{11}^{(j-1)}}{\partial t} &= -\frac{\partial}{\partial x}(c a_{11}^{(j-1)}) - \theta\,\frac{\partial a_{00}^{(j-1)}}{\partial x} - \frac{1}{\rho}\frac{\partial}{\partial x}(\rho a_{20}^{(j-1)}) \\
&\quad - \varepsilon\left[a_{00}^{(j-1)} q_1 + \frac{1}{\rho}\frac{\partial}{\partial x}(\rho a_{22}^{(j)})\right] \\
\frac{\partial a_{20}^{(j-1)}}{\partial t} &= -c\,\frac{\partial a_{20}^{(j-1)}}{\partial x} - \frac{2}{3}\theta\,\frac{\partial a_{11}^{(j-1)}}{\partial x} - \frac{2}{3}\frac{\partial c}{\partial x} a_{20}^{(j-1)} - \frac{\partial \theta}{\partial x} a_{11}^{(j-1)} \\
&\quad - \varepsilon\left[\frac{2}{3} a_{11}^{(j-1)} q_1 + a_{00}^{(j-1)} s_1 + \frac{2}{3}\frac{\partial c}{\partial x} a_{22}^{(j)} + \frac{5}{9\rho}\frac{\partial}{\partial x}(\rho a_{31}^{(j)})\right]
\end{aligned}
\tag{127}
$$

As the inductive assumption, we can take

$$a_{nl}^{(j-1)} = 0, \qquad (n, l) = (2, 2), (3, 1), \ldots \tag{128}$$

$$a_{nl}^{(j-i)} = 0, \qquad (n, l) = (0, 0), (1, 1), \ldots; \quad i = 2, 3, \ldots, j-1 \tag{129}$$

$$
\begin{aligned}
a_{n0}^{(j)} &= 0, \qquad n = 4, 6, \ldots \\
a_{n,1}^{(j)} &= -3\,\frac{\partial a_{20}^{(j-1)}}{\partial x}\,\mu_{31}^{n1}, \qquad n = 3, 5, \ldots \\
a_{n,2}^{(j)} &= -\frac{4}{3}\,\frac{\partial a_{11}^{(j-1)}}{\partial x}\,\mu_{22}^{n2}, \qquad n = 2, 4, 6, \ldots \\
a_{nl}^{(j)} &= 0, \qquad l = 3, 4, \ldots
\end{aligned}
\tag{130}
$$

after a sufficiently long time. Then Eqs. (127) may be reduced to

$$
\begin{aligned}
\frac{\partial a_{00}^{(j-1)}}{\partial t} &= -\frac{\partial a_{11}^{(j-1)}}{\partial x} \\
\frac{\partial a_{11}^{(j-1)}}{\partial t} &= -\frac{\partial a_{00}^{(j-1)}}{\partial x} - \frac{\partial a_{20}^{(j-1)}}{\partial x} + \frac{4\varepsilon}{3}\,\mu_{22}^{22}\,\frac{\partial^2 a_{11}^{(j-1)}}{\partial x^2} \\
\frac{\partial a_{20}^{(j-1)}}{\partial t} &= -\frac{2}{3}\,\frac{\partial a_{11}^{(j-1)}}{\partial x} + \frac{5\varepsilon}{3}\,\mu_{31}^{31}\,\frac{\partial^2 a_{20}^{(j-1)}}{\partial x^2}
\end{aligned}
\tag{131}
$$

which prescribe that $a_{00}^{(j-1)} = a_{11}^{(j-1)} = a_{20}^{(j-1)} = 0$ should be reached after a sufficiently long time. Combining this conclusion with Eqs. (128) and (129), we get

$$a_{nl}^{(j+1-i)}=0, \qquad (n,l)=(0,0),(1,1),...; \quad i=2,3,...,j \tag{132}$$

Therefore, the equation for $\xi^{(j+1)}$ is reduced to

$$I(\xi^{(j+1)})=\frac{4}{3}\frac{\partial a_{11}^{(j)}}{\partial x}e_{22}+3\frac{\partial a_{20}^{(j)}}{\partial x}e_{31} \tag{133}$$

It is sen from Eq. (133) that

$$\begin{aligned}
&a_{n,0}^{(j+1)}=0, \qquad n=4,6,...\\
&a_{n1}^{(j+1)}=-3\frac{\partial a_{20}^{(j)}}{\partial x}\mu_{31}^{n,1}, \qquad n=3,5,...\\
&a_{n2}^{(j+1)}=-\frac{4}{3}\frac{\partial a_{11}^{(j)}}{\partial x}\mu_{22}^{n,2}, \qquad n=2,4,...\\
&a_{nl}^{(j+1)}=0, \qquad l=3,4,...
\end{aligned} \tag{134}$$

and from Eq. (130) that

$$a_{nl}^{(j)}=0, \qquad (n,l)=(2,2),(3,1),... \tag{135}$$

It is evident from Eqs. (132), (134), and (135) that the proof has been accomplished.

It should be noticed that secular terms similar to Eq. (61) do not appear, although the linearized equations obtainable from Eq. (115) are the homogeneous equations corresponding to the linearized inhomogeneous ones of Eq. (119). In fact, for small vibrational variations to the solution of Eq. (115), ρ', c', $\theta' \sim e^{\lambda t+ikx}$, where $0>\operatorname{Re}\lambda \sim O(\varepsilon)$, and the order of magnitude of the inhomogeneous terms is $O(\varepsilon)$. By a discussion similar to that for Eq. (57), it can be seen that

$$\begin{pmatrix} a_{00}^{(1)} \\ a_{11}^{(1)} \\ a_{20}^{(1)} \end{pmatrix} \sim \begin{pmatrix} \rho_1 \\ c_1 \\ \theta_1 \end{pmatrix} \varepsilon t e^{\lambda t+ikx} \lesssim O(1)$$

which is different from Eq. (61), so these are not secular terms. Similarly, we have

$$\begin{pmatrix} a_{00}^{(j)} \\ a_{11}^{(j)} \\ a_{20}^{(j)} \end{pmatrix} \sim \frac{(\varepsilon t)^j}{j!} e^{\lambda t+ikx} \lesssim O(1)$$

and these are not secular terms either.

8. CONCLUDING REMARKS

In this paper we have shown that when the distribution function *and* the hydrodynamic variables are expanded in powers of the small Knudsen number ε, the requirement that no secular terms should appear causes the corresponding expansion of $\partial/\partial t$ to be terminated after $O(\varepsilon)$. Such a truncation was advocated by Cercignani on macroscopic grounds, since necessary boundary conditions for equations beyond Navier–Stokes cannot be established. In other words, we have shown Cercignani's truncation to be a consequence of the elimination of secular terms.

When the discussion is confined to normal solutions, as in the present paper, our method, in which the hydrodynamic variables are expanded in powers of ε, leaves some coefficients undetermined. Examples are $a_{nl}^{(j)}(x, 0)$ with $(n, l) = (0, 0)$, $(1, 1)$, $(2, 0)$ and $j = 1, 2,...$. This might be taken as a point in favor of Cercignani's alternative procedure, in which the hydrodynamic variables are expanded in powers of $\varepsilon^{N+1} = \varepsilon^2$. However, as will be shown in a companion paper,[15] this freedom is precisely what is needed when the normal solution is to be smoothly connected to a solution through the initial layer. In contrast, Cercignani's procedure would, at this point, generate yet another type of secular term.

Finally, both the Hilbert and the Enskog expansions may be used to discuss weak showk waves. The same discussion may be performed with the modified normal solution obtained in the present paper. The reason is that the shock wave problem is a steady one, and secular terms do not appear anyway. The same remark applies to the discussion of any steady transport process. In other words, our method leads to the same transport coefficients as the Enskog expansion. However, when one is concerned with relaxation processes, such as the *formation* of a shock wave, our method is superior.

The present method is different from the multi-time-scalling method, which is used by some authors to remove the secular terms in the normal solution under quite special conditions.[13]

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